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# GENUS IS SUPERADDITIVE UNDER BAND CONNECTED SUM

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DEFINITIONS. *Let* $k_1$, $k_0$ *be two knots in* $S^3$ *which can be separated by an embedded* 2-*sphere. Let* $b: I \times I \to S^3$ *be an embedding such that for* $i = 0, 1$ $f^{-1}(k_i) = I \times i$. $k = k_0 \#_b k_1 = (k_0 \cup k_1 \cup b(\partial I \times I)) - b(\mathring{I} \times \partial I)$ is called the *band connected sum* of $k_0$ and $k_1$. $b$ Is the *trivial band* and $k = k_0 \# k_1$ is a *connected sum* if there exists a two-sphere $Q$ separating $k_1$ from $k_0$ such that $Q \cap b(I \times I)$ is a connected arc. If $S_0$ and $S_1$ are disjoint Seifert surfaces for $k_0$ and $k_1$, disjoint from $b(I \times \mathring{I})$, then $S = S_0 \cup S_1 \cup b(I \times I)$ is called the *band connected sum* of $S_0$ and $S_1$.

$X$ is said to be obtained by *filling* the 3-manifold $M$ along the essential simple closed curve $\alpha \subset P$, $P$ a toral component of $\partial M$ if $X$ is obtained by first attaching a two-handle to $M$ along $\alpha$ and then capping off the resulting two-sphere with a 3-cell.

The purpose of this note is to prove the following result.

THEOREM 1. *If* $k$ *is a band connected sum of* $k_0$ *and* $k_1$, *then genus* $k \geq$ *genus* $k_1$ + *genus* $k_2$. *Equality holds if and only if there exists a Seifert surface for* $k$ *which is a band connected sum* (*using the same band*) *of minimal genus Seifert surfaces for* $k_1$ *and* $k_2$.

*Remarks*. This gives a positive proof of the twenty year old Tristram–Lickorish conjecture (see also problem 1, [2]), that genus is superadditive under band connected sum. In particular we obtain the main result of [3] "If the unknot is a band connected sum of $k_0$ and $k_1$, then each $k_i$ is the unknot and the band is the trivial band."

In [4] Thompson gave a new proof of the above mentioned result of Scharlemann. This paper is a refinement of that proof.

Theorem 1 has been independently proven, using different methods by Marty Scharlemann.

*Proof.* Let $D$ be an embedded disc in $S^3$ such that $D \cap (k \cup b(I \times I)) = b(I \times 1/2) \subset \mathring{D}$. Let $L = \partial D$. Let $S$ be an oriented surface of minimal genus in $S^3 - \mathring{N}(L)$ such that $\partial S = k$. We can assume that $S \cap D$ is an arc.

If $L$ bounds a disc disjoint from $S$, then $k$ is a connected sum of $k_0$ and $k_1$ and $b$ is the trivial band. In this case after finding a splitting two-sphere $Q$ and an $S$ such that $S \cap Q = S \cap D$ it is immediate that $S$ is a minimal genus surface for $k$ and is also a connected sum of minimal genus surfaces for $k_0$ and $k_1$, so Theorem 1 follows.

Now assume that $k \neq k_0 \# k_1$. Let $M = S^3 - \mathring{N}(k \cup L)$ and $P = \partial N(L)$. If there exists no non-boundary parallel torus $T \subset M$ separating $S$ from $P$, then $M$ is $S_P$-atoroidal (see Definition 1.6 [1]).

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If $M$ is not $S_P$-atoroidal, then $S$ is a minimal genus surface for $k$ in $S^3$. To see this let $T \subset M$ be a non-boundary parallel torus disjoint from $S$ which separates $P$ from $S$. $T$ can be isotoped in $M$ so that $T \cap D$ is a boundary parallel simple closed curve. $T$ is compressible in $W$, the closure of the component of $S^3 - T$ containing $k$, so $W$ is a solid torus. $W$ is knotted in $S^3$, else $T$ would be boundary parallel in $M$. Therefore, $T$ is incompressible in $S^3 - \mathring{W}$, so $S$ is a surface of minimal genus for $k$ in $S^3$.

We now show that if $M$ is $S_P$-atoroidal, then $S$ is a minimal genus surface for $k$ in $S^3$. By Corollary 2.4 [1], with at most one exception (up to isotopy), the following holds. If $N$ is obtained by filling $M$ along an essential simple closed curve in $P$, then $N$ is irreducible and genus $S \leq$ genus $S'$ if $S'$ is any embedded oriented surface in $N$ with $\partial S' = \partial S$. By Thompson [4] filling $M$ along the curve $\partial D$ yields a reducible manifold. [She observes that in $S^2 \times S^1$, the manifold obtained by performing 0–frame surgery to $L$, $k$ can be isotoped to $k_1 \# k_2$ by a series of applications of the "light bulb trick". So $M$ filled along $\partial D$ equals $S^2 \times S^1 \# (S^3 - \mathring{N}(k_1 \# k_2)$.] Therefore $\partial D$ must correspond to the exceptional filling and $S$ remains minimal genus when $M$ is filled along the meridian of $N(L)$.

Let $R$ be the surface $S - \mathring{N}(D)$. Note that $\partial R = k_0 \cup k_1$. Let $R'$ be a surface obtained by compressing $R$ a maximal number of times. Since $k_0$ and $k_1$ could be separated by a two sphere, $R'$ has two components, respectively spanning $k_0$ and $k_1$. We obtain

$$\text{genus } k = \text{genus } S = 1/2(2 - \chi(R)) \geq 1/2(2 - \chi(R')) \geq \text{genus } k_0 + \text{genus } k_1.$$

Therefore genus $k =$ genus $k_0 +$ genus $k_1$ if and only if $R$ is a union of minimal genus surfaces for $k_0$ and $k_1$, i.e. $S$ is a band connected sum of minimal genus surfaces for $k_0$ and $k_1$.

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